Def: An infinite series is an expression of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Note: There are two sequences to consider given a series $\sum_{n=1}^{\infty} a_n$.

(a) The sequence of terms / summands, a_n

(b) The sequence of partial sums, $S_k = \sum_{n=0}^{k} a_n$

$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2$$
$$S_3 = a_0 + a_1 + a_2 + a_3$$
$$\vdots$$
$$S_k = a_0 + a_1 + a_2 + \cdots + a_{k-1} + a_k$$

Def: If the sequence of partial sums converges to a real number S, then the infinite series converges. In this case, S is called the sum of the series, and we write

$$\sum_{n=0}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

* Do with $\frac{1}{2}$ first

Ex 1 Suppose r is a constant and consider the series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$. For which values of r does this series converge?

By HW: $\lim_{n\to\infty} r^n = 0 \longrightarrow |r| < 1$

Trick: $S_k = 1 + r + r^2 + \cdots + r^k$

$$(1-r)S_k = (1-r)(1 + r + r^2 + \cdots + r^k)$$

$$= 1 + r + r^2 + \cdots + r^k - r - r^2 - r^3 - \cdots - r^{k+1}$$

$$= 1 - r^{k+1}$$

$$\longrightarrow S_k = \frac{1-r^{k+1}}{1-r}$$

As $k \to \infty$, $r^{k+1} \to 0$

So $\sum_{n=0}^{\infty} r^n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r}$

$$= \frac{1}{1-r}$$

when $|r| < 1$

For $r > 1$, $\sum_{n=0}^{\infty} r^n = \infty$.

For $r < -1$, $\sum r^n$ DNE

Def: A geometric series is a series whose terms form a geometric seq.

Note: $\sum_{n=0}^{\infty} a r^n = a \sum_{n=0}^{\infty} r^n$

$= a \cdot \frac{1}{1-r} = \frac{a}{1-r}$

Ex 2 Find the exact value of the sum

$$10 - \frac{10}{9} + \frac{10}{81} - \frac{10}{729} + \frac{10}{6561} - \cdots$$

$10\left(\frac{1}{9}\right)^0 - 10\left(\frac{1}{9}\right)^1 + 10\left(\frac{1}{9}\right)^2 - 10\left(\frac{1}{9}\right)^3 + \cdots$

$10\left(\frac{-1}{9}\right)^0 + 10\left(\frac{-1}{9}\right)^1 + 10\left(\frac{-1}{9}\right)^2 + 10\left(\frac{-1}{9}\right)^3 + \cdots$

$$\sum_{n=0}^{\infty} 10\left(\frac{-1}{9}\right)^n = \frac{10}{1-\left(\frac{-1}{9}\right)} = \frac{10}{\frac{10}{9}}$$

$$= 9$$

Ex 3 Write the number $5.\overline{279} = 5.279279279\ldots$ as a ratio of two integers.

$$5 + \frac{279}{1000} + \frac{279}{1000000} + \frac{279}{1000000000} + \cdots$$

$$5 + 279 \cdot \left(\frac{1}{1000}\right)^1 + 279\left(\frac{1}{1000}\right)^2 + 279\left(\frac{1}{1000}\right)^3 + \cdots$$

$$5 + \sum_{n=1}^{\infty} 279 \cdot \left(\frac{1}{1000}\right)^n$$

$\left(\frac{1}{1000}\right)^0 + \left(\frac{1}{1000}\right)^1 + \cdots$

$$5 + \frac{279}{1000} \sum_{n=1}^{\infty} \left(\frac{1}{1000}\right)^{n-1}$$

$$5 + \frac{279}{1000}\left(\frac{1}{1-\frac{1}{1000}}\right) = 5 + \frac{279}{1000-1}$$

$$= 5 + \frac{279}{999} = \frac{4995+279}{999}$$

$$= 5274/999$$

Ex 4 The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic series*. Does it converge or diverge?

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Note: the terms $\to 0$

But: $1 + \underbrace{\frac{1}{2} + \frac{1}{3}}_{>\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{>\frac{1}{8}\text{ each}} + \cdots$

$\underbrace{\quad}_{>\frac{1}{2}} \quad \underbrace{\quad}_{>\frac{1}{2}} \quad \underbrace{\quad}_{>\frac{1}{2}}$

So $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots > \frac{1}{2} + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots}_{1}$

diverges to ∞

Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞

Thm: (Algebraic Properties of Convergent Series) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be Convergent series. Then the following algebraic properties hold:

(a) The series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(b) The series $\sum_{n=1}^{\infty} (a_n - b_n)$ converges and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

(c) For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$.

Ex 5 Does $\sum_{n=1}^{\infty}\left(2^{-\frac{1}{n}}-2^{-\frac{1}{n+1}}\right)$ converge or diverge? If it converges, to what does it converge?

$$\left(2^{-1}-2^{-\frac{1}{2}}\right)+\left(2^{-\frac{1}{2}}-2^{-\frac{1}{3}}\right)+\left(2^{-\frac{1}{3}}-2^{-\frac{1}{4}}\right)+\left(2^{-\frac{1}{4}}-2^{-\frac{1}{5}}\right)+\cdots$$

$$S_k=\left(2^{-1}-2^{-\frac{1}{2}}\right)+\cdots+\left(2^{-\frac{1}{k}}-2^{-\frac{1}{k+1}}\right)$$

$$=2^{-1}-2^{-\frac{1}{k+1}}$$

$$\sum_{n=1}^{\infty}2^{-\frac{1}{n}}-2^{-\frac{1}{n+1}}=\lim_{k\to\infty}S_k=\lim_{k\to\infty}2^{-1}-2^{-\frac{1}{k+1}}=2^{-1}-2^{0}$$

$$=\frac{1}{2}-1$$

$$=-\frac{1}{2}$$

Q. What are the partial sums of the series $\sum_{n=0}^{\infty}t_n-t_{n+1}$?

$$S_0=t_0-t_1$$

$$S_1=t_0-t_1+(t_1-t_2)=t_0-t_2$$

$$S_2=(t_0-t_1)+(t_1-t_2)+(t_2-t_3)=t_0-t_3$$

$$\vdots$$

$$S_k=t_0-t_k$$

$$\lim_{k\to\infty}S_k=\lim_{k\to\infty}t_0-t_k=t_0-\lim_{k\to\infty}t_k$$

if t_k converges

Def: Suppose that $(t_n)_{n=0}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty}t_n=L$. Then the series

$$\sum_{n=0}^{\infty}t_n-t_{n+1}$$

is called a telescoping series. It converges to t_0-L

Ex 6 Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

PFD: $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$1 = A(n+1) + Bn \rightarrow B = -1$, $A = 1$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots$

$t_n = \frac{1}{n}$ $t_{n+1} = \frac{1}{n+1}$

$= 1 - \lim_{n\to\infty} \frac{1}{n} = 1$

Q. How common are geometric and telescoping series? Will we be able to compute the values of very many sums?

Geometric and telescoping series are rare

Most sums are difficult or impossible to compute.

What can we do?

- Check for convergence or divergence
- understand fns. by relating them to near-geometric series.